# Construction of Haar measure

Let $G$ be a locally compact, topological group.

**Def** (Haar measure)
A left Haar measure on $G$ is a non-zero positive linear functional $m: C_c(G) \to \mathbb{C}$ that is invariant under left translation, that is such that $(g_* m)(f) = m(f)$ for all $f \in C_c(G)$
(or a Radon measure)

Most naive way: Define the measure on some compact set to be positive and use the group action to make it invariant.

Let $e \in U, K$ be compact subsets of $G$ with non-empty interior.

Notice that $K \subset \bigcup_{k \in K} kU$ and since $K$ is compact, there is a finite cover!

We denote by $h(K;U)$ the least number of translates of $U$ needed to cover $K$.

It holds that:

1) $h(K,K) = 1$
2) If $K' \subseteq K$, then $h(K',U) \leq h(K,U)$
3) for $K_1, K_2 \subset G$, $h(K_1 \cup K_2, U) \leq h(K_1, U) + h(K_2, U)$
4) $h(gK, U) = h(K, U)$
5) If $U$ is open, $K, B$ compact, then
$$h(K,U) \leq h(K,B)\, h(B,U)$$
6) If $(U_n)_n$ satisfies $U_{n+1} \subseteq U_n$ $\forall n$ and $\bigcap_n U_n = \{g\}$ and $K_1 \cap K_2 = \emptyset$, then: for large enough $n$
$$h(K_1 \cup K_2; U_n) = h(K_1; U_n) + h(K_2; U_n)$$

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Now we choose a "unit set" $K_0 \subseteq G$ (compact with non empty interior)

Further, we choose a sequence $(U_n)_n \subseteq G$ s.t. $\forall n$ $U_{n+1} \subseteq U_n$, $\bigcap_n U_n = \{g\}$

For any $K \subseteq G$ compact, non-empty interior we define $l_n(K) := \dfrac{h(K, U_n)}{h(K_0, U_n)}$

Clearly, $C_n(K) \neq 0 \ \forall n$

By 5) $\quad C_n(K) \leq h(K; K_0) \quad$ (replacing $B$ with $K_0$) $\quad \forall n$

Further, 5) with switching $K_0$ and $K$ gives $\quad C_n(K) \geq \dfrac{1}{h(K_0, K)} \quad \forall n$

Thus, there exists a converging subsequence $(n_i)$ with $\exists \lim\limits_{i\to\infty} C_{n_i}(K)$

Clearly, if $\quad \lim\limits_{i\to\infty} C_{n_i}(gK) = \lim\limits_{i\to\infty} C_{n_i}(K) \quad \forall g \in G$

For a countable family $K_m$ of compact sets with non-empty interior, we can, using a diagonal sequence argument, assume that

$$\forall m \quad \exists \lim_{i\to\infty} C_{n_i}(K_m)$$

Thus, we call the limit of $C_{n_i}(K)$ the content of $K$, if it exists.

Many details further down the road, one can prove:

If $K \subseteq G$ is a nice subset, i.e. $\partial K := \overline{K} - K^\circ$ is a null set

that is $\forall \varepsilon > 0 \ \exists$ compact non-empty int. set $K_\varepsilon$ s.t. $\limsup\limits_{i\to\infty} \dfrac{h(K_\varepsilon, U_{n_i})}{h(K, U_{n_i})} \leq \varepsilon$

then $\quad m(K) := \lim\limits_{i\to\infty} \dfrac{h(K; U_{n_i})}{h(K_0; U_{n_i})} \quad$ exists.

What can one observe directly.

1) $G$ compact iff $m < \infty$.

2) How did we get left-invariance?
By defining $h(K,U)$ as # of right $U$-translates $K = \bigcup_{k \in K} kU$.

If one defines it using $K = \bigcup_{k \in K} Uk$, one would get right invariance.

3) That the Haar measure is only well defined up to scalar!
(This has to do with choosing $K_0$ as "unit set")

4) $G$ is unimodular if left-covering number is "similar" to right covering number for all sets.